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93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2021
Calculus

Time allowed: Three hours
Total score: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Consider the function $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x - 3}$ where x is real.

Use a number line to show where $f(x) > 0$.

(b) Find ALL the real solutions of the following equation:

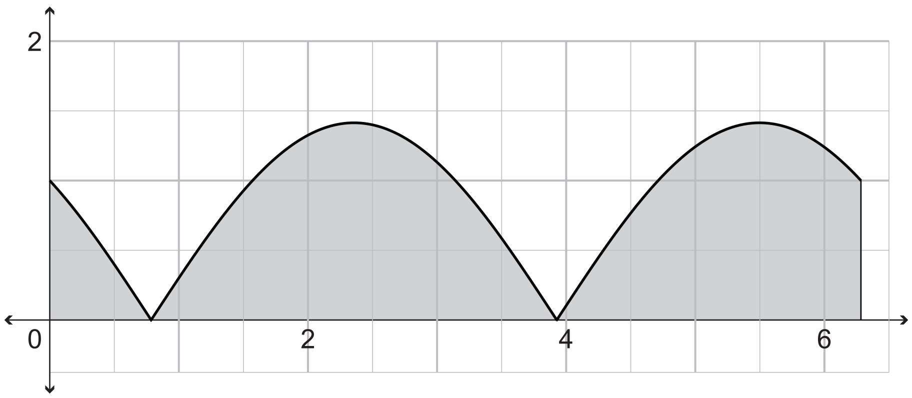
$$x^{x\sqrt{x}} = x^{2x}$$

(c) Find the equation of a vertical line which halves the area enclosed by:
$y = 2x^2 - x - 1$ and $y = -2x^2 - x + 1$

(d) Evaluate the definite integral:

$$\int_0^2 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$$

(e) The graph below shows the curve $y = |\sin x - \cos x|$ on the domain $0 \le x \le 2\pi$.



Find the exact value of the area between the curve, the x-axis, $x = 0$, and $x = 2\pi$.

QUESTION TWO

(a) If $\log_{\frac{a}{b}} b = 5$, where a and b are both positive, find the value of $\log_{\frac{a}{b}}\left(\sqrt[3]{b} \times \sqrt[4]{a}\right)$.

Note: the base of these logarithms is $\frac{a}{b}$.

(b) From all the possible pairs of positive rational numbers that add up to 11, find the pair that maximises the product of the square of the first number and the cube of the second number.

(c) Given $f(x) = a\sin(\pi x + \alpha) + b\cos(\pi x + \alpha) + 1$
and $f(2020) = 10$
find the value of $f(2021)$.

(d) Given $f(x) = \left(x^2 + 1\right)^{\sin x}$ find an exact value for $f'\left(\frac{\pi}{2}\right)$.

Hint: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln f(x)\right) = \frac{f'(x)}{f(x)}$

(e) Consider the function $f(x) = \left(\log_2 x\right)^2 + 6m\left(\log_2 x\right) + n$ where m and n are both real constants.

If the function has a local minimum at the point $\left(\frac{1}{8}, -2\right)$ what are the values of m and n?

QUESTION THREE

(a) The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\sin\theta$ and $\cos\theta$.

Show that:

$$\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta}=-\frac{b}{a}$$

(b) The line $y = mx + 2\sqrt{21}$ just touches the hyperbola $16x^2 - 9y^2 = 144$.

Find the value(s) of m.

(c) Consider the cubic function $f(x) = ax^3 - bx$ where a and b are both positive real constants.

$f(x)$ cuts the x-axis at the origin $(0,0)$ and also at the points P $\left(-\sqrt{3},0\right)$ and Q $\left(\sqrt{3},0\right)$.

Find the value of $f'(0)$ if the acute angle between the curve and the x-axis at points P and Q is equal to 45°.

(d) Five boys line up for a photo.

(i) In how many different ways can they be ordered in the line?

Two girls join in.

In how many different ways can the seven children be arranged if:

(ii) the two girls always stand next to each other

(iii) the two girls never stand next to each other.

QUESTION FOUR

(a) A young couple want to buy a house. They have \$76 000 in savings, which currently is insufficient for a mortgage deposit. After research, they invest all their capital in a Kiwisaver fund and commit to annual deposits of \$4500. The government will contribute \$500 to their fund each year as well. The fund has an anticipated average compounding growth rate of 16% per year.

The couple use the model $\frac{\mathrm{d}A}{\mathrm{d}t} = 0.16A + D$

to forecast the success of their plan where

- $A(t)$ is the amount of money in the investment at time t
- the deposit rate is \$$D$ per annum.

The couple hope to buy a house within 10 years, and believe they will need \$500 000 by then.

Will their initial investment of \$76 000 be sufficient, assuming everything else remains unchanged?

(b) The function $y = f(x)$ satisfies the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x-1)y^3$$

Given that $f(0) = a$ where a is a positive constant:

(i) Find an explicit expression for the function $y = f(x)$ in terms of a.

(ii) State the domain and range of the function in terms of a.

(iii) Draw a sketch of $y = f(x)$ as $a \to +\infty$ clearly labelling any key features.

(c) Consider the sequence defined by the formula:

$$T_{\mathrm{n}} = \sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}}$$

where $n = \{1, 2, 3, \ldots\}$.

Find the exact value of $\sum_{n=1}^{2021} T_{\mathrm{n}}$.

QUESTION FIVE

(a) Consider the points A (1,2) and B (3,4) on the Cartesian plane.
BC is a line perpendicular to the line AB.

If $|BC| = 4|AB|$ then find possible coordinates of C.

(b) $z = x + iy$, where x and y are real.

Find all complex solutions to the equation:

$$z + \frac{1}{\overline{z}} = \overline{z} + \frac{1}{z}$$

(c) Consider all the complex numbers z that satisfy the equation:

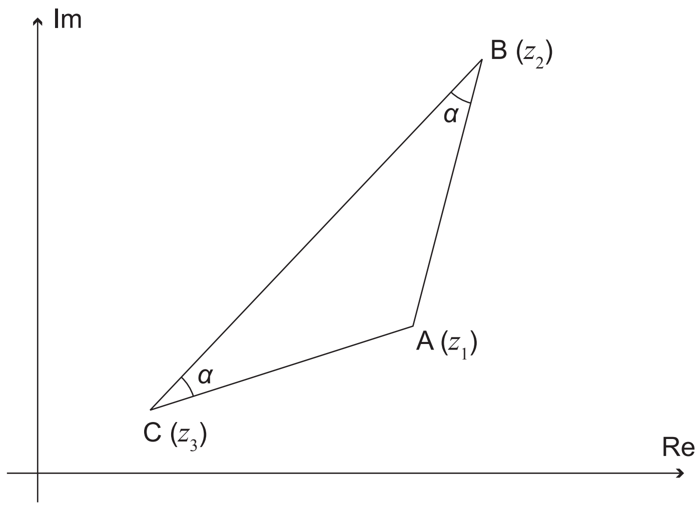
$$|z - (2 + 4i)| + |z - (3 + 6i)| = 4$$

(i) Sketch the locus of points that represent z on an Argand diagram.

(ii) Find the maximum value of $|z|$.

(d) ABC is an isosceles triangle with $\angle ABC = \angle ACB = \alpha$.

The vertices A, B, and C represent the complex numbers z_1, z_2, and z_3 respectively.



Show that: $(z_2 - z_1)(z_1 - z_3) = \left(\frac{1}{2}(z_2 - z_3)\sec\alpha\right)^2$

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